Topology of Metric Spaces

Definition of Open & Closed Balls

Definition Open Balls

Let (X,d) be a metric space. Then for any $x \in X$, and for any $r \in (0,\infty)$, the open ball centered at x of radius r is

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

Definition Closed Balls

Let (X,d) be a metric space. Then for any $x \in X$ and $r \in (0,\infty)$, the closed ball centered at x of radius r is

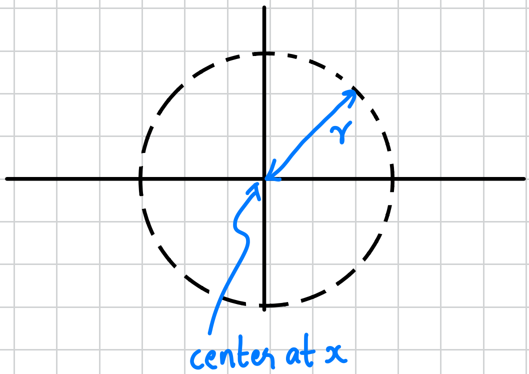
$$\overline{B}(x,r) = \{y \in X : d(x,y) \leq r\}$$

Examples of Open and Closed Balls

i) In $(\mathbb{R}^2, d_2)$,

$$d_2(x,y) = \sqrt{\sum_{i=1}^{2} |x_i - y_i|^2}$$

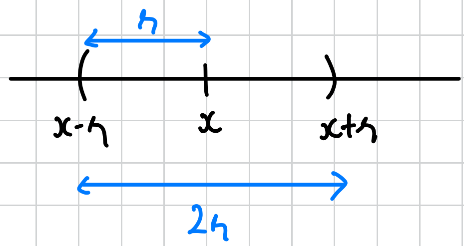
The open ball looks like the following : $B(x,r)$



iii) Classical example:
In metric $(\mathbb{R}, d_1)$

$$d_1(x,y) = |x-y|$$

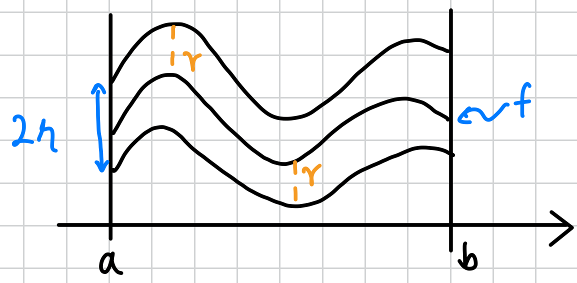
Open ball $B(x,r) = (x-r, x+r)$



ii) Consider the space $(C([a,b]), d_\infty)$

$$d_\infty(f,g) = \sup\{|f(x)-g(x)| : x \in [a,b]\}$$

The open ball $B(f,r)$ consists of all continuous functions $f' \in C([a,b])$ whose graphs lie whithin a band of vertical width of $2r$ centered at x_0



Neighbourhood

Definition: Neighbourhood

Let (X,d) be a metric space. A neighbourhood of the point $x \in X$ is any open ball in (X,d) with center x

Interior, Exterior and Boundary Points

Suppose that (X,d) is a metric space and $A \subseteq X$

Definition Interior Points

An interior point $y \in X$ of A is an element for which the open ball $B(y,\varepsilon)$ is contained entirely within A for some $\varepsilon > 0$

$$B(y,\varepsilon) \subseteq A$$

The interior of A is the set of all interior points, denoted by A°

$$\boxed{A^{\circ} = \text{set of all interior points}}$$

Definition Boundary points

The element $y \in X$ is a boundary point of A if and only if

$\forall\, \varepsilon > 0$, the open ball centered at y; $B(y,\varepsilon)$ hits the set and hits outside the set.

$$B(y,\varepsilon) \cap A \neq \emptyset \quad \text{and} \quad B(y,\varepsilon) \cap A^c \neq 0$$

The boundary of A is the set of all boundary points of A denoted by ∂A

$$\boxed{\partial A = \text{set of all boundary points}}$$

Definition Exterior Points

An element $y \in X$ is an exterior point of A if and only if

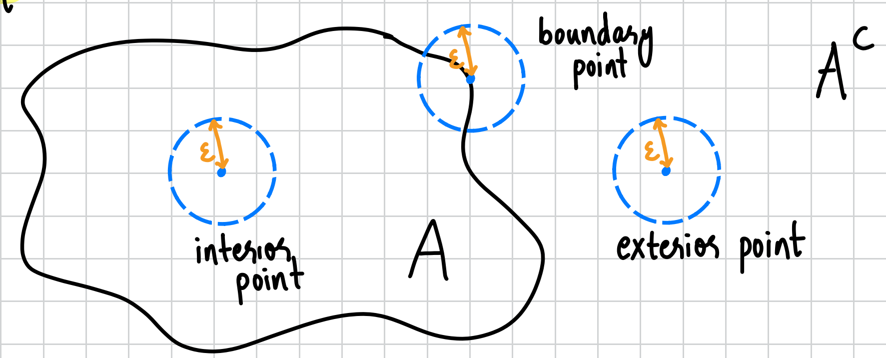
$\exists\, \varepsilon > 0$ for which

$$B(y,\varepsilon) \subseteq A^c$$

The exterior of A is the set of all exterior points denoted by A^e

$$\boxed{A^e = \text{set of all exterior points}}$$

Visualization:



Disjoint Union Property for Interior, Exterior and Boundary

Theorem Disjoint Unions

Let (X,d) be a metric space, and $A \subset X$. Then

$$X = \partial A \sqcup A^e \sqcup A^\circ$$

That is the following holds:

1) $A^e \cup A^\circ \cup \partial A = X$
2) $\partial A \cap A^\circ = \phi$
3) $\partial A \cap A^e = \phi$
4) $A^\circ \cap A^e = \phi$

Proof: Suppose $A \neq \phi$

1) Suppose $y \in X$ is an interior point. Then by defn of interior point $\exists\ \varepsilon > 0$ such that

$$B(y,\varepsilon) \subseteq A$$

And thus, we can draw the following conclusions:

(i) $B(y,\varepsilon) \subseteq A \Rightarrow B(y,\varepsilon) \nsubseteq A^c \Rightarrow y \notin A^c$

(ii) $B(y,\varepsilon) \subseteq A \Rightarrow B(y,\varepsilon) \nsubseteq A^c \Rightarrow B(y,\varepsilon) \cap A^c = \phi \Rightarrow y \notin \partial A$

And therefore $A^\circ \cap \partial A = \phi = A^\circ \cap A^e$

2) Suppose $y \in \partial A$. Then by definition of boundary, $\forall \varepsilon > 0$,

$$B(y,\varepsilon) \cap A \neq \phi \quad \text{and} \quad B(y,\varepsilon) \cap A^c \neq \phi$$

Thus $\nexists\ \varepsilon>0$ such that

$$B(y,\varepsilon)\subseteq A \Rightarrow y\notin A^{o} \quad \text{OR} \quad B(y,\varepsilon)\subseteq A^{c} \Rightarrow y\notin A^{e}$$

if $B(y,\varepsilon)\subseteq A$ then $B(y,\varepsilon)\cap A^{c}=\phi$ # if $B(y,\varepsilon)\subseteq A^{c}$ then $B(y,\varepsilon)\cap A=\phi$ #

Therefore $\partial A\cap A^{e}=\phi=\partial A\cap A^{o}$

3) Suppose $y\in A^{e}$. Then by the definition of exterior point,

$$B(y,\varepsilon)\subseteq A^{c}$$

(i) $B(y,\varepsilon)\subseteq A^{c} \Longrightarrow B(y,\varepsilon)\nsubseteq A^{c} \Rightarrow y\notin A^{o}$

(ii) $B(y,\varepsilon)\subseteq A^{c} \Rightarrow B(y,\varepsilon)\nsubseteq A^{c} \Rightarrow B(y,\varepsilon)\cap A=\phi \Rightarrow y\notin\partial A$

And therefore $A^{e}\cap\partial A=\phi=A^{e}\cap A^{o}$

Further, since for any $y\in X$, we can find $\varepsilon>0$ such that

1) $B(y,\varepsilon)\subseteq A \Rightarrow y\in A^{o}$
2) $B(y,\varepsilon)\subseteq A^{c} \Rightarrow y\in A^{c}$
3) $B(y,\varepsilon)\nsubseteq A$ and $B(y,\varepsilon)\nsubseteq A^{c} \Rightarrow y\in\partial A$

We have that

$$y\in A^{o}\cup\partial A\cup A^{e} \Rightarrow X\subseteq A^{o}\cup\partial A\cup A^{e}\subseteq X \qquad (*1)$$

Furthermore

$$A^{o}\cup\partial A\cup A^{e}\subseteq X \qquad (*2)$$

And therefore by principle of mutual containment $(*1)$ and $(*2)$ becomes

$$A^{o}\cup\partial A\cup A^{e}=X$$

Hence we have that

$$X=\partial A\sqcup A^{o}\sqcup A^{e} \qquad \blacksquare$$

Example Calculating Interior, Exterior and Boundary

Question

Calculate A^o, ∂A and A^e where A is the set

$$A = (0,1] \subseteq \mathbb{R}$$

with metric space $(\mathbb{R}, d)$ $d(x,y) = |x-y|$

Solution: Consider the following diagram of A

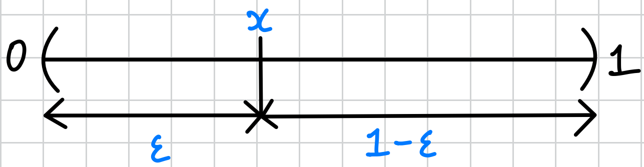
From the diagram above, we can deduce the following

- The interior: $A^o = (0,1)$
- The exterior: $A^e = (-\infty, 0) \cup (1, \infty)$
- The boundary: $\partial A = \{0, 1\}$

Claim 1

$$A^o = (0,1)$$

Proof: Take any $x \in (0,1)$. Then $0 < x < 1$



Let $\varepsilon = x - 0$, $\varepsilon' = 1 - x$ and take $\varepsilon^* = \min\{\varepsilon', \varepsilon\}$

Consider the open ball

$$B(y, \varepsilon^*/2)$$

Consider the point $y \in B(x, \varepsilon^*/2)$, then we get

$$y \in B(x, \varepsilon^*/2) \Rightarrow |y - x| < \frac{\varepsilon^*}{2}$$

$$\Rightarrow -\frac{\varepsilon^*}{2} < y - x < \frac{\varepsilon^*}{2}$$

$$\Rightarrow x - \frac{\varepsilon^*}{2} < y < x + \frac{\varepsilon^*}{2} \qquad (*1)$$

Now since $\varepsilon = x - 0$ and $\varepsilon^* = \min\{\varepsilon, \varepsilon'\} \Rightarrow \varepsilon^* \leq \varepsilon$

$$x - 0 = \varepsilon \Rightarrow x - \varepsilon = 0$$

$$\Rightarrow x - \varepsilon^* \geq 0 \qquad \text{since } \varepsilon^* \leq \varepsilon$$

$$\Rightarrow x - \frac{\varepsilon^*}{2} \geq 0$$

Similarly, since $\varepsilon' = 1 - x$ and $\varepsilon^* = \min\{\varepsilon', \varepsilon\} \Rightarrow \varepsilon^* \leq \varepsilon,$

$$\varepsilon' = 1 - x \Rightarrow 1 = \varepsilon' + x$$

$$\Rightarrow 1 \geq \varepsilon^* + x$$

$$\Rightarrow 1 \geq \frac{\varepsilon^*}{2} + x$$

And combining the results with $(*1)$, we get

$$0 \leq x - \frac{\varepsilon^*}{2} < y < x + \frac{\varepsilon^*}{2} \leq 1$$

and therefore all points $y \in B(x, \varepsilon^*/2)$ is an element of A, i.e.

$$B(x, \varepsilon^*/2) \subseteq A$$

Since by the definition, of an interior point, the definition implies the interior points points MUST belong to the set. Since points $x < 0$ and $x > 1$ lie outside the set, they are not interior points.

Further $1 \in A$ is not an interior point as for any $\varepsilon > 0$ and open ball $B(1, \varepsilon)$. This contains a point

$$y \in B(1, \varepsilon) \Rightarrow y > 1$$

$$\Rightarrow y \notin A$$

$$\Rightarrow B(1, \varepsilon) \nsubseteq A$$

Hence 1 is not an interior point, and therefore

$$A^{\circ} = (0, 1)$$

$\blacksquare$

Claim 2

$$\partial A = \{0, 1\}$$

Proof: First show that

$$\{0, 1\} \subseteq \partial A$$

and that no other point can be in $\partial A \Rightarrow \partial A = \{0, 1\}$

Showing for $0 \in \partial A$ (similar for 1), consider open ball $B(0,\varepsilon)$

$$B(0,\varepsilon) = (-\varepsilon,\varepsilon), \quad \varepsilon > 0.$$

Clearly since $(-\varepsilon,\varepsilon) \cap A \neq \emptyset$ and $(-\varepsilon,\varepsilon) \cap A^c \neq \emptyset \Rightarrow 0$ is a boundary point $\Rightarrow 0 \in \partial A$

Now we must show there are no other boundary points. By the disjoint union property,

$$\partial A \cap A^{\circ} = \emptyset$$

Therefore no point in A° is a boundary point $\Rightarrow (0,1) \not\subset \partial A$

Now consider the point $y > 1$. Then $\exists\, \varepsilon > 0$ such that $y = 1+\varepsilon$. And then take the open ball "say"

$$B(y, \varepsilon/10)$$

Since $\varepsilon/10 < \varepsilon$, the open ball $B(y, \varepsilon/10)$ will not intersect A since it is more than $\varepsilon/10$ away from 1. Therefore

$$B(y, \varepsilon/10) \subseteq A^c \Rightarrow B(y,\varepsilon) \cap A = \emptyset$$

Similar argument for $y < 0$

■

Claim 3

$$A^{e} = (-\infty, 0) \cup (1, \infty)$$

Proof: By the disjoint union property

$$\partial A \cup A^{\circ} \cup A^{e} = X$$

$$\partial A \cap A^{\circ} \neq \emptyset$$

$$A^{\circ} \cap A^{e} \neq \emptyset$$

$$\partial A \cap A^{e} \neq \emptyset$$

Then

$$A^{e} = \mathbb{R} \setminus (A^{\circ} \cup \partial A) = A^{e} = (-\infty, 0) \cup (1, \infty)$$

■

Open and Closed Sets

Suppose (X,d) is a metric space and $A \subseteq X$

Definition Open Sets

A subset A of X is open if and only if

$$A \cap \partial A = \emptyset$$

Definition Closed Sets

A subset $A \subseteq X$ is closed if and only if

$$\partial A \subseteq A$$

Note: Open is <u>NOT</u> the opposite of closed

Definition Clopen sets

A set that is both open AND closed is called a clopen set

Properties of Open and Closed Sets

Theorem $\partial A = \partial A^c$

Suppose (X,d) is a metric space and $A \subseteq X$. Then

$$\partial A = \partial A^c$$

<u>Proof</u>: By the definition of an boundary point, $y \in \partial A$ if and only if $\forall$ open balls B centered at y,

$$B(y,\varepsilon) \cap A^c \neq \emptyset \quad \text{and} \quad B(y,\varepsilon) \cap A^c \neq \emptyset \qquad (1)$$

By the definition of complement of a set

$$(A^c)^c = A$$

And therefore for any open balls centered at y, (1) becomes

$$B(y,\varepsilon) \cap A^c \neq \emptyset \quad \text{and} \quad B(y,\varepsilon) \cap (A^c)^c \neq \emptyset$$

and this is the definition of boundary of A^c: ∂A^c

Therefore ALL points in ∂A are points in ∂A^c and vice-versa and hence

$$\boxed{\partial A = \partial A^c}$$

$\blacksquare$

Theorem A is open if and only if A^c is closed

Let (X,d) be a metric space and let $A \subseteq X$ be a open set. Then

$$A \text{ is open} \iff A^c \text{ is closed}$$

Proof:

($\Rightarrow$): Lets assume that $A \subseteq X$ is open. Then $A=\emptyset$ or $A=\emptyset$

1) CASE 1: $A=\emptyset$

If $A=\emptyset$, then by definition of complement of of a set,

$$A^c = X$$

Since X is clopen $\Rightarrow$ X is closed

2) CASE 2: $A \neq \emptyset$

If $A \neq \emptyset$, then by definition,

$$\partial A \cap A = \emptyset \Rightarrow \partial A \subseteq A^c$$

Note $\partial A = \partial A^c$

Therefore

$$\partial A \subseteq A^c \Rightarrow \partial A^c \subseteq A^c$$

which by the definition of closed: A^c is closed.

($\Leftarrow$): Suppose that A^c is closed. Then by the definition of closed

$$\partial A^c \subseteq A^c \Rightarrow \partial A^c \cap A = \emptyset$$

Note $\partial A = \partial A^c$

Therefore

$$\partial A^c \cap A = \emptyset \Rightarrow \partial A \cap A = \emptyset$$

and by the definition of open: A is open.

$\blacksquare$

A is open $\Rightarrow$ no boundary points

$\Rightarrow \exists\, \varepsilon > 0$ s.t $B(x,\varepsilon) \subseteq A$ or $B(x,\varepsilon) \subseteq A^c$

$\Rightarrow$ x is an interior OR x is an exterior

Theorem X and ϕ are clopen

Let (X,d) be a metric space. Then

$$\phi \text{ and } X \text{ are clopen sets}$$

Proof:

Claim 1

$$\boxed{\phi \text{ is clopen}}$$

Proof: The boundary of an empty set is empty.

$$\boxed{\partial\phi = \phi}$$

And therefore by definition of open sets

$$\phi \cap \partial\phi = \phi \cap \phi = \phi$$

which shows ϕ is open.

Further by the definition of closed sets,

$$\partial\phi = \phi \subseteq \phi$$

(since the empty set is the subset of ALL sets including itself)

and therefore ϕ is closed.

$\blacksquare$

Claim 2

$$\boxed{X \text{ is clopen}}$$

Proof: From boundary of the whole space X is empty, i.e.

$$\boxed{\partial X = \phi}$$

and hence we can deduce that

$$\partial X \cap X = \phi \cap X = \phi$$

which shows X is open.

Further, by definition of closed sets

$$\partial X = \phi \subseteq X$$

(since ϕ is the subset of ALL sets)

and therefore closed. $\blacksquare$

Examples: Examples of open and closed sets

1) Consider the sets

$$(0,1),\quad [0,1],\quad (0,1]$$

Here

- The set $(0,1)$ is open
- The set $[0,1]$ is closed
- The set $(0,1]$ neither open nor closed

2) Consider the set

$$A=(0,1)\cup(1,2)$$

The boundary $\partial A=\{0,1,2\}$. Further

$$\partial A\cap A=\emptyset \Rightarrow A \text{ is open}$$

$$\partial A\nsubseteq A \Rightarrow A \text{ is not closed}$$

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Open and Closed Sets on Subspaces

On subspaces, we need to be careful.

Let (X,d) be a metric space and $A\subseteq X$ be a subspace

$$(A,d)$$

Note Since A is a subspace, it cannot see what points lie outside A

Therefore it can only see points that lie in A, i.e.

$$\boxed{A^c=\emptyset}$$

Example: Open and Closed on subspaces.

1) Consider the metric space $(\mathbb{R},d)$ and consider the subspace

$$(A,d|_A)$$

where

$$A=(0,1)\cup(1,2)\subseteq\mathbb{R}$$

therefore $B_A(x,r)\cap A^c=\emptyset$

i) The boundary of A are: $\boxed{\partial A=\emptyset}$ as $1,2,0\notin A$ and subspace cannot see $\mathbb{R}$

as $A^c=\emptyset$
$\Rightarrow \partial A=\emptyset$

ii) Here, the interior is the entire set, i.e.

$$A^{\circ} = A$$

iii) Since the subspace cannot see beyond A, there are NO exterior points,

$$A^{e} = \emptyset$$

2) Now consider the subset

$$(0,1) \subseteq (0,1) \cup (1,2)$$

The subset is only contained in $A = (0,1) \cup (1,2)$, not in $\mathbb{R}$ since A is a subspace.

Open Sets using Interior Points (equivalent defn)

Let (X,d) be a metric space and $A \subseteq X$. If A is open, then every point of A is an interior point.

Theorem Open sets using Interior points

Let (X,d) be a metric space and $A \subseteq X$. Then

$$A \text{ is open} \iff \forall x \in A, \exists \varepsilon = \varepsilon(x) \text{ such that } B(x,\varepsilon) \subseteq A$$

<u>Proof</u>: $A = \emptyset$ or $A \neq \emptyset$

$(\Rightarrow)$:

1) CASE 1: $A = \emptyset$

Since the empty set $\emptyset$ is clopen $\Rightarrow$ $\emptyset$ is open

2) CASE 2: $A \neq \emptyset$

Since A is open, it will have no boundary point, and therefore it will satisfy the contrapositive (negation) of the definition of boundary

$$\exists \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subseteq A \quad \text{OR} \quad B(x,\varepsilon) \subseteq A^c \qquad \forall x \in A$$

Since $x \in A$, $B(x,\varepsilon) \nsubseteq A^c$ otherwise $x \in A^c$ which is a contradiction. Thus by elimination

$$B(x,\varepsilon) \subseteq A$$

$(\Leftarrow)$: Assume that $\forall x \in A, \exists \varepsilon = \varepsilon(x) > 0$ such that

$$B(x,\varepsilon) \subseteq A \qquad (*)$$

As a consequence

$$x \notin \partial A$$

since if $x \in \partial A$, we can find $\varepsilon > 0$ such that $B(x,\varepsilon) \cap A^c \neq \emptyset$ which contradicts (*)

$$x \notin \partial A \Rightarrow \partial A \cap A = \emptyset$$
$$\Rightarrow A \text{ is open.}$$

$\blacksquare$

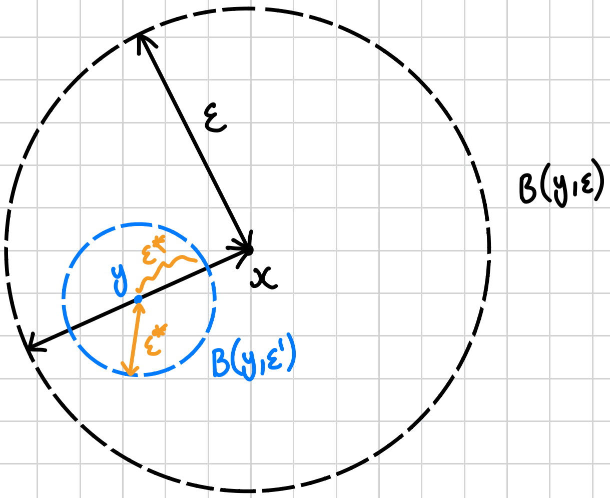
Open Ball is an Open Set

Theorem Open Ball is an open set

In any metric space (X,d),

Open ball is an open set

Proof: Consider the following diagram



Take the open ball centered at x with radius ε. Then consider

$$y \in B(x,\varepsilon)$$

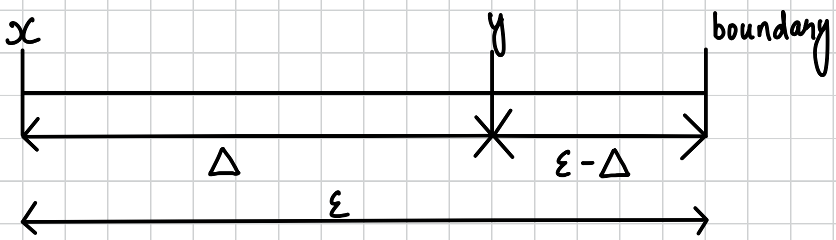
1) If $y = x$, then we are done as consider open ball $B(y,\varepsilon)$

$$B(y,\varepsilon) = B(x,\varepsilon) \subseteq B(x,\varepsilon) \qquad \text{(sets contain themselves)}$$

Therefore by definition, $B(x,\varepsilon)$ is open.

2) If $y \neq x$, then by axiom (M1) of metric spaces,

$$d(x,y) = \Delta > 0 \qquad \text{and} \qquad 0 < \Delta < \varepsilon$$



So we let $\varepsilon' = \min\{\Delta, \varepsilon - \Delta\}$ and let $\varepsilon^* = \dfrac{\varepsilon'}{2}$

Claim

$$B(y,\varepsilon^*) \subseteq B(x,\varepsilon)$$

That is by definition of open ball we want to show that

$$\forall z \in B(y,\varepsilon^*) \Rightarrow d(x,z) < \varepsilon \quad \Rightarrow z \in B(x,\varepsilon)$$

By (M4), triangle inequality of metrics,

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &\leq \Delta + \varepsilon^* \qquad \text{as } z \in B(y,\varepsilon^*) \\ &= \Delta + \min\{\Delta, \varepsilon-\Delta\}/2 \\ &\leq \Delta + \frac{\varepsilon-\Delta}{2} \\ &< \Delta + \varepsilon - \Delta \\ &= \varepsilon \end{aligned}$$

$$\Rightarrow z \in B(x,\varepsilon) \quad \forall z \in B(y,\varepsilon^*)$$

And therefore we have that

$$B(y,\varepsilon^*) \subseteq B(x,\varepsilon)$$

and by definition $B(x,\varepsilon)$ is an open set

$\blacksquare$

Singleton sets

Proposition Singleton sets are closed

Let (X,d) be a metric space. Then,

$$\{x_0\} \subseteq X \text{ is closed}$$

Proof: Here, either $X=\{x_0\}$ or X contains at least 2 points

1) CASE 1: $X=\{x_0\}$

The entire metric space $X=\{x_0\}$ is clopen $\Rightarrow \{x_0\}$ is closed

2) CASE 2: X contains at least 2 points

X consists of atleast 2 points $\Rightarrow \{x_0\}^c \neq \phi$

Let $y \in \{x_0\}^c$.

$y \neq x_0 \Rightarrow d(x_0, y) = \varepsilon > 0$ for some $\varepsilon > 0$

Then we have that

$$B(y, \varepsilon/2) \subseteq \{x_0\}^c \Rightarrow \{x_0\}^c \text{ is open}$$

$$\Rightarrow \{x_0\} \text{ is closed}$$ A^c open $\Leftrightarrow$ A closed

■

$\boxed{\text{Singletons are always closed}}$

Singletons in discrete metric space : d_0

Consider discrete metric space (X, d_0)

$$d_0(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Now all open balls in (X, d_0) are of form

$$\boxed{B(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\} = \begin{cases} \{x_0\} & 0 < \varepsilon \leq 1 \\ X & \varepsilon > 1 \end{cases}}$$

So for $\varepsilon \leq 1$, $B(x_0, \varepsilon) = \{x_0\}$ and open ball is an open set

$\Rightarrow$ $\boxed{\{x_0\} \text{ is open in discrete metric}}$

$\Rightarrow$ $\boxed{\text{singletons open in discrete metric space}}$

As shown above, singletons are closed

$\boxed{\text{Therefore for discrete metric space, singletons are clopen}}$

Topology on Metric Spaces

Definition Topology on (X,d)

The topology of a metric space (X,d) is the collection of all open subsets of X

$$T_d = \{\Omega \subseteq X : \Omega \text{ is open}\}$$

Therefore

$$\boxed{\Omega \in T_d \Leftrightarrow \Omega \text{ is open}}$$

Properties of Topology

Property T1: Closed under Arbitrary Unions

Theorem Closed Under Arbitrary Unions

Take any collection of open sets, say

$$\Lambda \subseteq T_d$$

Then the union of open sets is open.

i.e. $\forall \Lambda \subseteq T_d$,

$$\bigcup_{\Omega \in \Lambda} \Omega \in T_d \quad \text{(it is open)}$$

Proof: Take any

$$x \in \bigcup_{\Omega \in \Lambda} \Omega$$

Then

$$\exists\, x \in \Omega(x) \qquad \text{AND} \qquad \Omega \text{ is an open set}$$

And therefore by the definition of open

$$\exists\, \varepsilon(x) > 0 \quad \text{such that} \quad B(x,\varepsilon) \subseteq \Omega$$

and hence we have

$$B(x,\varepsilon) \subseteq \Omega \quad \text{AND} \quad \Omega \subseteq \bigcup_{\Omega \in \Lambda} \Omega$$

Then by Transitivity of subsets,

$$B(x,\varepsilon) \subseteq \bigcup_{\Omega \in \Lambda} \Omega$$

and therefore by definition, open.

$\blacksquare$

Property T2: Closed under Finite Collections of Open Sets

Theorem Closed Under Finite Intersections

Take any finite collection of open sets

$$\Omega_1, \ldots, \Omega_N$$

Then

$$\bigcap_{i=1}^{N} \Omega_i \text{ is open } (\in T_d)$$

Proof: $\bigcap_{i=1}^{N} \Omega_i$ can be empty or non-empty

1) CASE 1: If $\bigcap_{i=1}^{N} \Omega_i = \emptyset$ then as $\emptyset$ is open $\Rightarrow \emptyset \in T_d$

2) CASE 2: If $\bigcap_{i=1}^{N} \Omega_i \neq \emptyset$ then take any

$$x \in \bigcap_{i=1}^{N} \Omega_i$$

Since Ω_i is open,

$$\exists\, \varepsilon_i > 0 \text{ such that } B(x, \varepsilon_i) \subseteq \Omega_i$$

Take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$ then

$$B(x,\varepsilon) \subseteq B(x,\varepsilon_i) \quad \forall\, i \in \{1, \ldots, N\} \implies B(x,\varepsilon) \subseteq \bigcap_{i=1}^{N} \Omega_i$$

$\blacksquare$

Note $X, \emptyset \in T_d$

Remark The intersection of infinite number of open sets need not be open.
To see a counterexample, let

$$B_n = B(0, \tfrac{1}{n}) \subseteq \mathbb{R}^2, \quad 1, 2, \ldots$$

where B_n is an open ball in the complex plane and open ball is an open set in $\mathbb{R}^2$
However

$$\bigcap_{i=1}^{\infty} B_n = \{0\}$$

which is not open since $\nexists$ no open ball in the complex plane with center 0 that is contained in 0

Open set is a Union of Open Balls

Theorem Open set is the union of open balls.

Suppose (X,d) is a metric space and $A \subseteq X$. Then

$$A \text{ is open} \iff A \text{ is a union of open balls contained in } A$$

$$A = \bigcup_{x \in A} B(x,\varepsilon)$$

Proof: Take any $x \in A$ and therefore

$$\forall x \in A \ \exists \varepsilon \in \varepsilon(x) \text{ such that } B(x,\varepsilon) \subseteq A$$

Now since $x \in B(x,\varepsilon(x))$ and therefore if we go through all points $x \in A$,

$$A \subseteq \bigcup_{x \in A} B(x,\varepsilon(x)) \qquad (*1)$$

Further, we can see that since $B(x,\varepsilon(x)) \subseteq A$ for any x,

$$\bigcup_{x \in A} B(x,\varepsilon(x)) \subseteq A$$

Therefore by combining $(*1)$ and $(*2)$ and using mutual containment

$$A = \bigcup_{x \in A} B(x,\varepsilon) \qquad \blacksquare$$

Theorem Let (X,d) be a metric space and $A \subseteq X$. Then

i) A° is an open subset of A that contains every open subset of A

ii) A is open $\iff A = A^\circ$

That is A° is the biggest open subset of A

Proof:

i) Let $x \in A^\circ$. By definition of interior point

$$\exists \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subseteq A$$

But $B(x,\varepsilon)$ is an open set and therefore

$$\forall y \in B(x,\varepsilon) \ \exists \varepsilon^* > 0 \text{ such that } B(y,\varepsilon^*) \subseteq B(x,\varepsilon) \subseteq A$$

Therefore all points of $B(x,\varepsilon)$ is an interior point of A therefore

$$B(x,\varepsilon) \subseteq A^{\circ}$$

Thus x is a center of an open ball contained in A° and this is true for any $x \in A^{\circ}$.

Therefore A° is open.

We need to show A° contains all open subsets $G \subseteq A$. To show this
Let $x \in G$.

$$G \text{ is open} \Rightarrow \forall x \in G \ \exists \varepsilon > 0 \text{ s.t } B(x,\varepsilon) \subseteq G \subseteq A$$

$$\Rightarrow x \text{ is an interior point of } A$$

$$\Rightarrow x \in A^{\circ}$$

Therefore we have $x \in G \Rightarrow x \in A^{\circ}$

$\Rightarrow G \subseteq A^{\circ}$ for any arbitrary open set G

ii) A is open then $A \subseteq A^{\circ}$. We also have by defn $A^{\circ} \subseteq A$. Hence $A = A^{\circ}$ ■

Collection of Closed Sets

Lets see what happens to the properties of Topology of metric spaces when we replace open sets with closed sets

Definition Collections of Closed Sets $\mathcal{F}_e$

Take an arbitrary collection of closed sets of X and let it be denoted by

$$\mathcal{F}_e = \{F \subseteq X : F \text{ is closed}\}$$

Properties of $\mathcal{F}_e$

Property F1: Closed under Arbitrary Intersections

Theorem Closed under Arbitrary Intersections

Take any collection of closed sets

$$\Lambda \subseteq \mathcal{F}_e$$

The intersection of closed sets is closed

i.e. $\forall \Lambda \subseteq \mathcal{F}_e$

$$\bigcap_{F \in \Lambda} F \in \mathcal{F}_e \quad \text{(it is closed)}$$

Proof: As proven earlier, A is open $\iff A^c$ is closed. Therefore

$$\forall F \in \mathcal{F}_e \Rightarrow F^c \in T_d$$

$$\Rightarrow \bigcup_{F \in \Lambda} F^c \in T_d \qquad \text{Property T1}$$

Now by De Morgan's Law

$$\bigcup_{F \in \Lambda} F^c = \left(\bigcap_{F \in \Lambda} F^c\right)^c$$

and therefore

$$\bigcup_{F \in \Lambda} F^c \in T_d \Rightarrow \left(\bigcap_{F \in \Lambda} F\right) \in \mathcal{F}_e$$

where T_d is the topology on metric spaces. And hence $\bigcap F$ is closed. ∎

Property T2: The union of collection of closed sets

Theorem The Union of Finite Collection of Closed Sets

Take any finite collection of closed sets

$$F_1, \dots, F_N$$

Then

$$\bigcap_{i=1}^{N} F_i \quad \text{is closed} \quad (\in \mathcal{F}_e)$$

Closure of a Set

Make any set closed by adding boundary points

- Anything less will not work
- Anything more will be too much

Definition Closure

Let (X,d) be a metric space and $A \subseteq X$. The closure of A is

$$\bar{A} = A \cup \partial A$$

Note: A may be both open or closed

Note $A \subseteq \bar{A}$, $\bar{A}$ is a superset. $\bar{A}$ is the smallest superset of A to be closed

$$\boxed{A \subseteq \bar{A} \text{ and } A \text{ is closed} \Rightarrow A = \bar{A}}$$

Closure is Closed

Theorem $\bar{A}$ is closed

The closure of A which is $\bar{A}$ is closed, i.e.

$$\bar{A} = A \cup \partial A \quad \text{is closed}$$

Proof: It is sufficient to show that $(\bar{A})^c$ is open.

1) CASE 1: If $(\bar{A})^c = \phi$, ϕ is clopen so we are done

2) CASE 2: If $(\bar{A})^c \neq \phi$, then we need to show that (defn of open)

$$\forall x \in (\bar{A})^c, \exists \varepsilon = \varepsilon(x) \text{ such that } B(x,\varepsilon) \subseteq (\bar{A})^c$$

Take $x \in (\bar{A})^c$ and we get the following

$$\begin{aligned} x \in (\bar{A})^c &\Rightarrow x \notin \bar{A} \\ &\Rightarrow x \notin (A \cup \partial A) \\ &\Rightarrow x \notin A \text{ and } x \notin \partial A \end{aligned}$$

Since $x \notin \partial A$, by negation of definition of boundary

$$x \notin \partial A \Rightarrow \exists \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subseteq A \text{ or } B(x,\varepsilon) \subseteq A^c$$

Since $x \notin A$, $B(x,\varepsilon) \nsubseteq A$ and therefore by cancellation,

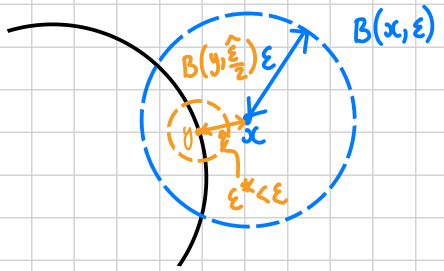
$$B(x,\varepsilon) \subseteq A^c$$

But we also need to stay away from ∂A

Suppose
$$\exists\ y \in B(x,\varepsilon) \text{ such that } y \in \partial A \qquad \text{(leads to a contradiction)}$$

Then $y \in \partial A$ and by definition of boundary,
$$B(y,\delta) \cap A \neq \emptyset \quad \text{and} \quad B(y,\delta) \cap A^c \neq \emptyset$$

Consider the following diagram:



Let $d(x,y) = \varepsilon^* < \varepsilon$

Consider the open ball $B(y, \hat{\varepsilon}/2)$ and define
$$\hat{\varepsilon} = \min\{\varepsilon^*, \varepsilon - \varepsilon^*\}$$

Thus we get $B(y,\hat{\varepsilon}) \subseteq B(x,\varepsilon)$ and $B(x,\varepsilon) \subseteq A^c \Rightarrow \boxed{B(y,\hat{\varepsilon}) \subseteq A^c}$ (*) $\Rightarrow B(y,\hat{\varepsilon}) \cap A = \emptyset$

But we have that $B(y,\hat{\varepsilon}) \cap A \neq \emptyset$ and this contradicts (*) (defn of boundary)

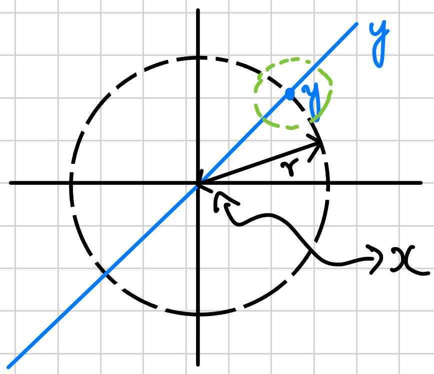
$$\therefore y \notin \partial A$$

■

Example of Closure of an Open Ball

Consider metric space $(\mathbb{R}^2, d_2)$ where
$$d_2(x,y) = \sqrt{\sum_{i=1}^{2} |x_i - y_i|}$$

Showing that in $(\mathbb{R}^2, d_2)$,
$$\overline{B(x,r)} = \underbrace{\bar{B}(x,r)}_{\text{closed ball}}$$



Observation: no point in
$$B(x,r) = \{y \in \mathbb{R}^2 : d_2(x,y) < r\}$$
is a boundary point. Indeed they are all interior point

(Recall ∂A, A° and A^e are pairwise disjoint)

Therefore it is sufficient to show that

$$\partial B(x,r) = \{y \in \mathbb{R}^n : d_2(x,y) = r\} \subseteq \bar{B}(x,r)$$

Define $\bar{B}(x,r) := C$

Since all points in $B(x,r)$ is an interior point, any $y \in \mathbb{R}^2$ for which $d_2(x,y) > r$ is an exterior of $B(x,r)$

We know that any $\underline{r} \in L_y$ can be written in the form

$$\underline{r}(\lambda) = \underline{x} + \lambda(y - \underline{x}) \qquad \lambda \in \mathbb{R}$$

i) If $-1 < \lambda < 1$, then $\underline{r} \in B(x,r)$

ii) If $\lambda \notin (-1,1)$ then $r \notin B(x,r)$

iii) If $\lambda = 1$ then $\underline{r}(1) = \underline{x} + 1(y - \underline{x}) = \underline{y}$

Let $\varepsilon > 0$ be given. Choose $\delta := \min\left\{\frac{\varepsilon}{2}, \frac{r}{2}\right\}$. Then

$$\underline{r}(\delta) = \underline{x} + \delta(\underline{y} - \underline{x}) \in B(x,r)$$

&

$$\underline{r}(1+\delta) = \underline{x} + (1+\delta)(\underline{y} - \underline{x}) \in B(x,r)^c$$

From this, we can deduce that

$$\left.\begin{array}{l} B(y,\delta) \cap B(x,r) \neq \phi \\ B(y,\delta) \cap B(x,r) \neq \phi \end{array}\right\} \text{Definition of boundary}$$

Note that $\bar{B}(x,r) = B(x,r) \cup C$

◀

Note This is not true in general.

$\exists (X,d)$ such that $\overline{B(x,r)} \neq \bar{B}(x,r)$

Theorem

Let (X,d) be a metric space and $A \subseteq X$.

$$A \text{ is closed} \iff A = \bar{A}$$

Proof:

i) Suppose $\bar{A} = A$.

Closure $\bar{A}$ is closed $\Rightarrow$ A is closed

ii) Suppose A is closed

A is closed $\Rightarrow \partial A \subseteq A$

$\Rightarrow \bar{A} = A = A \cup \partial A$

■

Limit Points of a Set

Motivation: We want to form an "efficient" closing of a set. That is we want

$\boxed{\text{Suppose } A \subseteq F \subseteq \bar{A} \text{ and } F \text{ is closed} \Rightarrow F = \bar{A}}$

That is $\boxed{\bar{A} \text{ is the smallest closed superset of } A}$

Definition Limit Point

Let (X,d) be a metric space and $F \subseteq X$. A point $x \in X$ is called a limit point of F if each open ball with center x contains at least one point of F different from x, i.e.

$$B(x,\varepsilon) \setminus \{x\} \cap F \neq \emptyset \qquad \text{for any } \varepsilon > 0$$

The derived set of A denoted by A' is the set of ALL limit points of A

$$\boxed{A' = \text{set of all limit point}}$$

Proposition

Let (X,d) be a metric space and $F \subseteq X$. If x_0 is a limit point of F, then every open ball $B(x_0,r)$ contains an infinite number of points of F

↳ $B(x_0,r) \cap F$ has infinite points

Proof: (via contradiction)

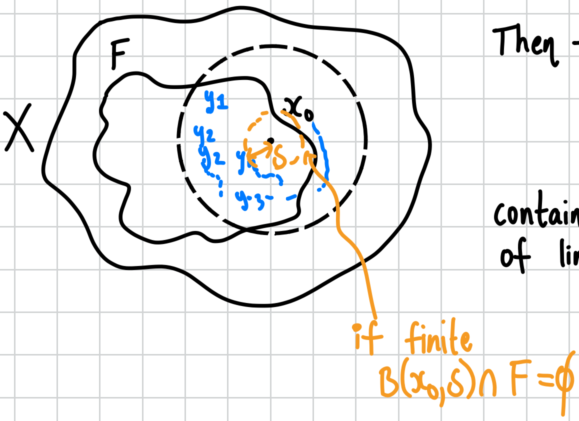
Suppose that the ball

$$B(x_0,r)$$

contains a finite number of points of F. Therefore consider the following set of points:

$$y_1,\dots,y_N \in B(x_0,r) \cap F \qquad y_i \neq x_0$$

and define $\delta = \min\{d(y_1,x_0), d(y_2,x_0), \dots, d(y_N,x_0)\}$



Then the open ball

$$B(x_0,\delta) \qquad \text{i.e. } B(x_0,\delta) \cap F = \emptyset$$

contains no points of F distinct from x_0 contradicting the defn of limit point ⨳

■

Proposition

Let (X,d) be a metric space and $F \subseteq X$.

x_0 is a limit point of $F \iff \exists$ a sequence $(x_n) \in F^{\mathbb{N}}$ of distinct points such that

$$\lim_{n\to\infty} d(x_n, x_0) = 0$$

Proof:

$(\Leftarrow)$: Suppose $\exists\ (x_n) \in F^{\mathbb{N}}$ such that $\lim_{n\to\infty} (x_n, x_0) = 0$

Then every open ball $B(x_0, r)$ contains $(x_n)_{n \geq N_0}$ for a suitable choice of N_0.

Observe that i) Finitely many points do not affect limit

ii) Points $x_1, x_2, \dots, x_n, \dots \in F \Rightarrow B(x_0, r)$ contains a point of F different from x_0

$\Rightarrow x_0$ is a limit point by defn.

$(\Rightarrow)$: Suppose x_0 is a limit point of F.

Choose a point $x_1 \in F$ such that

$$x_1 \in B(x_0, 1) \quad \text{and} \quad x_1 \neq x_0$$

Further, choose a point $x_2 \in F$ such that

$$x_2 \in B(x_0, 1/2) \quad \text{and} \quad x_2 \neq x_1 \neq x_0$$

Possible by previous proposition

Continuing this process in which the n^{th} step of the process is a point $x_n \in F$ such that

$$x_n \in B(x_0, 1/n) \quad \text{and} \quad x_n \neq x_0 \neq x_1 \cdots \neq x_{n-1}$$

And in the limit $n \to \infty$, we have a sequence (x_n) of distinct points of F such that

$$\lim_{n\to\infty} d(x_n, x_0) = 0 \quad \blacksquare$$

$$y \in F' \iff y \text{ is a limit point}$$
$$\iff \exists\, y' \in F \text{ such that } y' \neq y \text{ and } y' \in B(y, \varepsilon)$$
$$\Rightarrow B(y, \varepsilon) \cap F \neq \phi$$

Equivalent ways of writing limit point

Equivalent definition for closed

Theorem Closed

Let (X,d) be a metric space and $F \subseteq X$.

If $F' \subseteq F$ (F contains all its limit points) $\Rightarrow$ F is closed.

Proof: (using contradiction)

Suppose F is closed and $\exists y$ such that $y \in F'$ and $y \notin F$

Let $\varepsilon > 0$ be given and consider open ball $B(y, \varepsilon)$

i) $\exists y' \in F$ such that $y' \neq y$ and $y' \in B(y, \varepsilon)$ (since y is a limit point)

$$\Rightarrow B(y,\varepsilon) \cap F \neq \emptyset$$

ii) $B(y,\varepsilon) \ni y$ and $y \notin F \Rightarrow B(y,\varepsilon) \cap F^c \neq \emptyset$

From (i) and (ii), y is a boundary point of F $\Rightarrow y \in \partial F$ and $\partial F \subseteq F$ (F is closed)

$$\Rightarrow y \in F$$

This is a contradiction. ※ ■

Closure using Limit Points

We can also form a closure by adding all the limit points

$\boxed{\textbf{Fact} \quad \bar{A} = A \cup A'}$

Showing that this is equivalent to the first form of closure, $(\bar{A} = A \cup \partial A)$

i) If A is closed $\Rightarrow \bar{A} = A$ and $A' \subseteq A \Rightarrow A' \subseteq \bar{A} \Rightarrow \bar{A} = A \cup A'$

ii) If A is not closed, we know that

$$\bar{A} = A \cup \partial A$$

If $A' \neq \emptyset$ then $A' \subseteq \bar{A}$ (empty set is the subset of all sets)

Therefore assume that $A' \neq \emptyset$ and further assume that $\underbrace{y \in A' \text{ and } y \notin A}$

if $y \in A'$ and $y \notin A$ then $y \in \bar{A}$ as $A \subseteq \bar{A}$

Since $y \notin A \Rightarrow y \in \partial A$. By defn of boundary

$\forall \varepsilon > 0$, $B(y,\varepsilon) \cap A \neq \emptyset$ and because $y \notin A \Rightarrow \exists y' \neq y$ s.t $y' \in B(y,\varepsilon) \cap A$

$\Rightarrow y$ is a limit point

Therefore we have that $y \in \bar{A}$ and $y \notin A \Rightarrow y \in \partial A$

$\Rightarrow y \in A'$

and thus $A' \subseteq \bar{A}$

Since $A \subseteq \bar{A}$ and $A' \subseteq \bar{A} \Rightarrow \boxed{A' \cup A \subseteq \bar{A}}$ $(*1)$

Now showing that $\bar{A} \subseteq A \cup A'$: As A is not closed, $\exists\, y \in A'$ such that $y \notin A$ (if A is closed $A' \subseteq A$)

Let $\varepsilon > 0$ be given. Then

i) y is a limit point, $\exists\, y' \neq y \in A$ s.t $y' \in B(y, \varepsilon) \Rightarrow B(y,\varepsilon) \cap A \neq \phi$

ii) $y \notin A \Rightarrow y \in A^c \Rightarrow B(y,\varepsilon) \cap A^c \neq \phi$

Therefore by (i) and (ii) $\Rightarrow y$ is a boundary point.

Hence we have that $y \in A \cup A'$ and $y \notin A \Rightarrow y \in A'$

$\Rightarrow y$ is a boundary.

Thus $\partial A \subseteq A \cup A'$

Since $A \subseteq A \cup A'$ and $\partial A \subseteq A \cup A' \Rightarrow \boxed{\bar{A} \subseteq A \cup A'}$ $(*2)$

Therefore by $(*1)$ and $(*2)$ and by principle of mutual containment

$$\bar{A} = A \cup A'$$

$\blacksquare$

Remark In the proof above, we used the following equivalence

$$\boxed{A \subseteq B \cup C \iff A \cap B^c \subseteq C}$$

This follows from the logical equivalence of

$$\boxed{p \to q \vee r \equiv p \wedge \sim q \to r}$$

Note It is **not** the case

$$A' = \partial A$$

They could be **different**

Topological Version of Closure

Theorem Topological version of closure

Let (X,d) be a metric space and $A \subseteq X$. Then

$$\bar{A} = \bigcap_{\mathcal{F}} F$$

where $\mathcal{F}$ is the collection of all closed supersets of A

Showing that $A \subseteq F \subseteq \bar{A}$ and F is closed $\Rightarrow F = \bar{A}$

Proof: Let $A \subseteq X$

If $A \subseteq F \subsetneq \bar{A}$ where F is closed, then F contains all its limit point but not all of A's limit points.

otherwise $A' \subseteq F \subseteq \bar{A} \Rightarrow F = \bar{A}$

Suppose that $y \in A'$ and $y \notin F$

By defn of limit point, for any $\varepsilon > 0$, $B(y,\varepsilon) \cap A \neq \emptyset$ and contains $x \neq y$.

But $x \in A \subseteq F \Rightarrow x \in F$ and y is a limit point of A

$\Rightarrow y \in F'$

But F is closed $\Rightarrow$ F is closed

$\Rightarrow F' \subseteq F$

$\Rightarrow y \in F$ #

■

Derived set is closed

Proposition The derived set is closed

Let (X,d) be a metric space. Then F' is closed, i.e.

$$(F')' \subseteq F'$$

Proof: Either $(F')' = \emptyset$ or $(F')' \neq \emptyset$

1) CASE 1: $(F')' = \emptyset$ and $\emptyset \subseteq F' \Rightarrow (F')' \subseteq F$

2) CASE 2: $(F')' \neq \emptyset$

Suppose $x_0 \in (F')'$ and consider open ball $B(x_0, r)$

By definition of limit point,

$$\exists\, y \neq x_0 \text{ and } y \in F' \text{ s.t } y \in B(x_0, r)$$

Define $r' = r - d(x_0, y) \Rightarrow r' < r$.

$y \in F' \Rightarrow y$ is a limit point

$\Rightarrow B(y, r')$ contains an infinite number of points of F

But $B(y, r') \subseteq B(x_0, r)$ (look at proof of open ball is an open set)

$\Rightarrow B(x_0, r)$ contain infinitely many points

$\Rightarrow x_0$ is a limit point

$\Rightarrow x_0 \in F'$

Therefore we have shown that

$\boxed{(F')' \subseteq F'} \Rightarrow$ set of all limit points is closed.

$\blacksquare$

Properties of Derived Set

Theorem Properties of Derived sets

Let (X, d) be a metric space and $F_1 \subseteq X$ and $F_2 \subseteq X$. Then

i) $F_1 \subseteq F_2 \Rightarrow F_1' \subseteq F_2'$

ii) $(F_1 \cup F_2)' = F_1' \cup F_2'$

iii) $(F_1 \cap F_2)' \subseteq F_1' \cap F_2'$

Proof:

i) Suppose $x \in F_1' \Rightarrow x$ is a limit point of F_1

$\Rightarrow \exists\, y \neq x \in F_1$ s.t $y \in B(x, \varepsilon)$

Now, since, $F_1 \subseteq F_2 \Rightarrow y \in F_2$

$\Rightarrow y \in B(x, \varepsilon)$ and $y \neq x$ with $y \in F_2$

$\Rightarrow y$ is a limit point of F_2

$\Rightarrow y \in F_2'$.

Therefore $F_1' \subseteq F_2'$

ii) $F_1 \subseteq F_1 \cup F_2 \Rightarrow F_1' \subseteq (F_1 \cup F_2)'$ from (i) (*1)

$F_2 \subseteq F_1 \cup F_2 \Rightarrow F_2' \subseteq (F_1 \cup F_2)'$ from (i) (*2)

Therefore by (*1) and (*2),

$$F_1' \cup F_2' \subseteq (F_1 \cup F_2)' \qquad (*1)$$

Now suppose $x_0 \in (F_1 \cup F_2)'$.

Then $\exists$ a sequence $(x_n)_{n=1}^{\infty}$ of distinct points in $F_1 \cup F_2$ s.t

$$d(x_n, x_0) \to 0 \text{ as } n \to \infty$$

Since (x_n) is an infinite sequence, one of F_1 or F_2 contains an infinite number of points of x_n (subsequence, all distinct)

a) if F_1 contains an infinite number of points of x_n, then

$$x_0 \in F_1' \text{ and } F_1' \subseteq F_1' \cup F_2' \Rightarrow x_0 \in F_1' \cup F_2'$$

subsequence converges to same limit

b) if F_2 contains an infinite number of points of x_n, then

$$x_0 \in F_2' \text{ and } F_2' \subseteq F_1' \cup F_2' \Rightarrow x_0 \in F_1' \cup F_2'$$

subsequence converges to same limit

Therefore from a) and b),

$$(F_1 \cup F_2)' \subseteq F_1' \cup F_2' \qquad (*2)$$

From (*1) and (*2), and principle of mutual containment,

$$(F_1 \cup F_2)' = F_1' \cup F_2'$$

iii) Similar to first part of (ii)

$\blacksquare$

Theorem: Equivalence of Closure

Let (X,d) be a metric space, $F \subseteq X$. Then the following statements are equivalent

i) $x \in \overline{F}$

ii) $B(x,\varepsilon) \cap F \neq \emptyset$ for every open ball centered at x

(iii) $\exists$ an infinite sequence (x_n) of points (not necessarily distinct) of F such that

$$x_n \to x \text{ as } n \to \infty \quad \left(\lim_{n\to\infty} d(x, x_n) = 0\right)$$

Note that

$$A \subseteq \bar{A} \text{ and } B \subseteq \bar{B} \Rightarrow A \cup B \subseteq \bar{A} \cup \bar{B}$$

But closure is the smallest closed set

$$\Rightarrow \boxed{\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}}$$

Theorem: Subset of Closure.

Suppose (X,d) is a metric space and $A, B \subseteq X$.

$$A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$$

Proof:

Suppose $x \in A$

$$x \in \bar{A} \Rightarrow \underset{(1)}{x \in A} \text{ or } \underset{(2)}{x \in A'}$$

(1) $x \in A \Rightarrow x \in B$ (hypothesis)

$\Rightarrow x \in \bar{B}$ $\quad B \subseteq \bar{B}$

(2) $x \in A' \Rightarrow x$ is a limit point

$\Rightarrow \exists\, y \neq x$ s.t $y \in A$ and $y \in B(x,\varepsilon)$

$\Rightarrow \exists\, y \neq x$ s.t $y \in B$ and $y \in B(x,\varepsilon)$

$\Rightarrow B(x,\varepsilon) \setminus \{x\} \cap B \neq \emptyset$

$\Rightarrow y \in B'$

Properties of Interior, Exterior and Boundary

The following are properties of interior, exterior and boundary:

Theorem A^e is open

Let (X,d) be a metric space and let $A \subseteq X$. Then

$$A^e \text{ is open.}$$

Proof:

Any point $y \in X$ is an exterior point of A iff $\exists\, \varepsilon > 0$ such that $B(y,\varepsilon) \subseteq A^c$

$\{y' \in X : d(y,y') < \varepsilon\}$ complement

1) CASE 1: $A^e = \emptyset$, as $\emptyset$ is clopen $\Rightarrow \emptyset$ is open.

2) CASE 2: $A^e \neq \emptyset$.

For any $x \in A^e$, by definition of exterior,

$$\exists\, \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subseteq A^c$$

Take $\varepsilon^* < \varepsilon$, $\forall y \in B(x,\varepsilon)$, $\exists\, \varepsilon^*$ s.t $B(y,\varepsilon^*) \subseteq B(x,\varepsilon) \subseteq A^c$ $*$ (open balls are open)

Therefore all points of $B(x,\varepsilon)$ are exterior points by $*$, hence

$$B(x,\varepsilon) \subseteq A^e$$

This is the definition of open $\Rightarrow A^e$ is open. $\blacksquare$

Theorem: ∂A is closed.

Let (X,d) be a metric space and $A \subseteq X$. Then

$$\partial A \text{ is closed}$$

Proof: Using the disjoint union property,

$$X = A^\circ \sqcup A^e \sqcup \partial A$$

$A^\circ \cap \partial A = \emptyset$

$A^\circ \cap A^e = \emptyset$

$\partial A \cap A^e = \emptyset$

Therefore we have that

$$\partial A = X \setminus (A^\circ \sqcup A^e)$$

Further since A° and A^e are open,

$$A^\circ \sqcup A^e \in \mathcal{T}_d \Rightarrow A^e \sqcup A^\circ \text{ is open.}$$ (union of open sets are open)

But $\partial A = (A^\circ \sqcup A^e)^c \Rightarrow \partial A$ is closed (A is closed $\Leftrightarrow A^c$ is open). $\blacksquare$

Theorem $\partial(\partial A) \subseteq \partial A$

Let (X,d) be a metric space and $A \subseteq X$. Then
$$\partial(\partial A) \subseteq \partial A$$

Proof:

Since ∂A is closed,
$$\partial(\partial A) \subseteq \partial A$$
$\blacksquare$

Equivalent definition of Interior

Theorem $A^\circ = A \setminus \partial A$

Let (X,d) be a metric space and $A \subseteq X$. Then
$$A^\circ = A \setminus \partial A$$

Proof:

(i) Showing that $A^\circ \subseteq A \setminus \partial A$

If $A^\circ = \phi$ then $A^\circ \subseteq A$ is trivially true

If $A^\circ \neq \phi$ then suppose $x \in A^\circ$. We know that $A^\circ \subseteq A$

We must show that no interior point is a boundary point.

$x \in A^\circ \Rightarrow \exists\ \varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$. (defn of interior point)

if $x \in \partial A$, then $B(x,\varepsilon) \cap A \neq \phi$ and $B(x,\varepsilon) \cap A^c \neq \phi$

contradicts the fact that $B(x,\varepsilon) \subseteq A$

$\Rightarrow x \notin \partial A$

Therefore $x \in A^\circ \Rightarrow x \notin \partial A$

$\Rightarrow A^\circ = A \setminus \partial A \Rightarrow x \in A \setminus \partial A$

$\Rightarrow A^\circ \subseteq A \setminus \partial A$

ii) Showing that $A \setminus \partial A \subseteq A^\circ$

If $A \setminus \partial A = \phi$ then $A \setminus \partial A \subseteq A^\circ$ is trivially true

If $A \setminus \partial A \neq \phi$ then suppose $x \in A \setminus \partial A$.

$x \in A \setminus \partial A \Rightarrow x \in A$ and $y \notin \partial A$

And by negation of definition of boundary;

$$x \notin \partial A \Rightarrow B(x,\varepsilon) \subseteq A \text{ or } B(x,\varepsilon) \subseteq A^c$$ not true as $x \in A$

since $x \in A$, $B(x,\varepsilon) \nsubseteq A^c$ and therefore

$$B(x,\varepsilon) \subseteq A \Rightarrow x \text{ is an interior point}$$

$$\Rightarrow x \in A^\circ$$

Therefore by (i) and (ii),

$$A^\circ = A \setminus \partial A$$

■

Equivalent definition of Exterior

It can be shown that the exterior is the complement of closure

Theorem Exterior is the complement of closure

Let (X,d) be a metric space and $A \subseteq X$. Then

$$A^e = (\bar{A})^c$$

Proof: First observe that

$$x \in (\bar{A})^c \Leftrightarrow x \notin A \text{ and } x \notin \partial A$$

(i) Showing that $A^e \subseteq (\bar{A})^c$

$$x \in A^e \Rightarrow \exists \varepsilon > 0 \text{ s.t } B(x,\varepsilon) \subseteq A^c$$ (defn of interior)

if $x \in \partial A$ then, $B(x,\varepsilon) \cap A \neq \phi \Rightarrow$ contradiction

$$\left.\begin{array}{r} \Rightarrow x \notin \partial A \\ \text{Further } B(x,\varepsilon) \subseteq A^c \Rightarrow x \in A^c \Rightarrow x \notin A \end{array}\right\} \Rightarrow x \notin (A \cup \partial A)$$

$$\Rightarrow x \in (\bar{A})^c$$

$$\Rightarrow A^e = (\bar{A})^c$$

(ii) Showing that $(\bar{A})^c \subseteq A^e$

Suppose $x \in (\bar{A})^c \Rightarrow x \notin A$ and $x \notin \partial A$

$x \notin A \Rightarrow x \in A^c$ and by negation of defn of boundary,

not true as $x \notin A$ $\quad B(x,\varepsilon) \subseteq A$ and $B(x,\varepsilon) \subseteq A^c$.

But since $x \in A^c$, $B(x,\varepsilon) \subseteq A^c \Rightarrow x \in A^e$

■

Union of Boundary

Theorem Boundary of union is contained in union of boundary

Let (X,d) be a metric space and $A,B \subseteq X$. Then,

$$\partial(A \cup B) \subseteq \partial A \cup \partial B$$

Proof:

Suppose $x \in \partial(A \cup B)$. By definition of boundary,

$$\forall \varepsilon > 0, \quad B(x,\varepsilon) \cap (A \cup B) \neq \phi \quad \text{and} \quad B(x,\varepsilon) \cap (A \cup B)^c \neq \phi$$

By De-Morgan's Law, we get

$$B(x,\varepsilon) \cap (A \cup B) \neq \phi \quad \text{and} \quad B(x,\varepsilon) \cap (A^c \cap B^c) \neq \phi$$

$$\Rightarrow \quad (B(x,\varepsilon) \cap A) \cup (B(x,\varepsilon) \cap B) \neq \phi \quad \text{and} \quad B(x,\varepsilon) \cap A^c \cap B^c \neq \phi$$

$(A \cup B \neq \phi \Rightarrow A \neq \phi \text{ or } B \neq \phi)$

$$\Rightarrow \quad (B(x,\varepsilon) \cap A) \neq \phi \text{ or } B(x,\varepsilon) \cap B \neq \phi \quad \text{and} \quad B(x,\varepsilon) \cap A^c \cap B^c \neq \phi$$

$$\Rightarrow \quad B(x,\varepsilon) \cap A \neq \phi \text{ and } B(x,\varepsilon) \cap A^c \cap B^c \neq \phi \quad \text{or} \quad B(x,\varepsilon) \cap B \neq \phi \text{ and } B(x,\varepsilon) \cap A^c \cap B^c \neq \phi$$

$$\Rightarrow \quad \underbrace{B(x,\varepsilon) \cap A \neq \phi \text{ and } B(x,\varepsilon) \cap A \neq \phi}_{\partial A} \quad \text{or} \quad \underbrace{B(x,\varepsilon) \cap B \neq \phi \text{ and } B(x,\varepsilon) \cap B^c \neq \phi}_{\partial B}$$

if $B(x,\varepsilon) \cap A = \phi$ then $B(x,\varepsilon) \cap A^c \cap B^c = \phi$

$$\Rightarrow \quad x \in \partial A \quad \text{or} \quad x \in \partial B$$

$$\Rightarrow \quad x \in \partial A \cup \partial B$$

∎

Theorem $A^e = A^c \setminus (\partial A^c)^c$

Let (X,d) be a metric space and $A \subseteq X$. Then,

$$A^e = A^c \setminus (\partial A^c)^c$$

Proof: We know that

$$
\begin{aligned}
A^e &= (\bar{A})^c \\
&= (A \cup \partial A)^c && \text{(De Morgan's law)} \\
&= A^c \cap (\partial A)^c \\
&= A^c \cap (\partial A^c)^c && (\partial A = \partial A^c)
\end{aligned}
$$

$$\Rightarrow A^e = A^c \setminus (\partial A^c)^c$$

∎

Some other useful properties are

$$\boxed{\begin{aligned} \partial A &\subseteq \partial(A \cup B) \\ \partial B &\subseteq \partial(A \cup B) \end{aligned}}$$

Note if a set is closed,

$$\boxed{A = \bar{A} \implies A^e = A^c}$$

Density

Motivation:

$\mathbb{Q} \subseteq \mathbb{R}$

rationals — reals

Take any $x \in \mathbb{R}$ and any $\varepsilon > 0$. $\exists\, p/q \in \mathbb{Q}$ such that

$$|x - p/q| < \varepsilon$$

$$\boxed{\overline{\mathbb{Q}} = \mathbb{R}} \implies \boxed{\mathbb{Q}' = \mathbb{R}}$$

Definition: Density

Let (X,d) be metric spaces and $A \subseteq X$.

A is said to be dense in $X \Leftrightarrow \bar{A} = X$

That is for any $x \in X$ and any $\varepsilon > 0$, $\exists\, \alpha \in A$ such that

$$d(x, \alpha) < \varepsilon$$

Topological Equivalence

Definition: Topological Equivalence

Let (X,d) and (X,d^*) be metric spaces. Then (X,d) and (X,d^*) are equivalent if

$$T_d = T_{d^*}$$

That is if the metrics d and d^* generate the same open sets

Theorem: Let X be a set and d^* and d be metrics on X such that $\exists\, \lambda > 0$ for which

$$\frac{1}{\lambda} d(x,y) \leq d^*(x,y) \leq \lambda d(x,y)$$

Then

$$T_d = T_{d^*}$$

Proof: Since open sets are union of open balls,

Take an open set in (X,d^*), i.e. $\Omega \in T_{d^*}$. Then it is the union of open balls

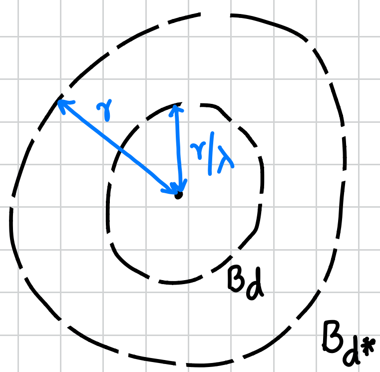
$$\Omega = \bigcup_{x \in \Omega} B_{d^*}(x,r)$$

Suppose

$$B_d(x,r) = \{y \in X : d(x,y) < r\} \qquad B_{d^*}(x,r) = \{y \in X : d^*(x,y) < r\}$$

Also, we know that $B_d(x, r/\lambda) \subseteq B_{d^*}(x,r)$ as

$$\begin{aligned} y \in B_d(x,r/\lambda) &\Rightarrow d(x,y) < \frac{r}{\lambda} \\ &\Rightarrow \lambda d(x,y) < r \\ &\Rightarrow d^*(x,y) < r \\ &\Rightarrow y \in B_{d^*}(x,r) \end{aligned}$$



Now, since $B_d(x,r/\lambda) \subseteq B_{d^*}(x,r)$, we can say that

$$B_d(x,r/\lambda) \subseteq \Omega \qquad \text{for any } x \in X$$

And hence by defintion of open, Ω is open in d metric. Therefore

$$\Omega \in T_d$$

(similar for other way)

Subspaces

Open and Closed do not behave "well" in subspaces

Let (X,d) be a metric space, $A \subseteq X$ and $(A, d|_A)$ be a subspace. Then

1) **Open sets** in A as a subspace are all of the form

$$A \cap \Omega \quad \text{where} \quad \Omega \in T_d$$

2) **Closed sets** are of the form

$$A \cap F \quad \text{where} \quad F^c \in T_d$$